# Bethe Ansatz and Thermodynamic Limit of Affine Quantum Group Invariant Extensions of the t-J Model[1]

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## Abstract

We have constructed a one dimensional exactly solvable model, which is based on the t-J model of strongly correlated electrons, but which has additional quantum group symmetry, ensuring the degeneration of states. We use Bethe Ansatz technique to investigate this model. The thermodynamic limit of the model is considered and equations for different density functions written down. These equations demonstrate that the additional colour degrees of freedom of the model behave as in a gauge theory, namely an arbitrary distribution of colour indices over particles leave invariant the energy of the ground state and the excitations. The $S$-matrix of the model is shown to be the product of the ordinary $t - J$ model $S$-matrix and the unity matrix in the colour space.

# 1 Introduction

Since the discovery of high-$T_c$ cuprate superconductivity the one dimensional physics of strongly correlated electrons has been in focus in many publications [1]. The Hubbard [2] and $t - J$ models [3] are such examples , motivated in part by the fact that high-$T_c$ compounds display antiferromagnetism in the absence of doping. The $t - J$ model was proposed by Zhang and Rice [3] and describes strongly correlated electrons with anntiferromagnetic exchange interaction.

The interest in one dimensional physics grew after Andersons claim [4] that two-dimensional systems may have features in common with one dimensional systems. In addition it should be mentioned that powerful methods in 1D such as bosonization, 2D conformal field theory and in particular the Bethe ansatz technique allow the detailed study of such systems.

The $t - J$ model may be used as well for heavy fermion system [5, 6].

At the supersymmetric point $J = 2t$ the $t - J$ model becomes exactly integrable [5, 7, 8, 9, 10, 11, 12] because the Hamiltonian can be represented as a graded permutation in a superalgebra of two fermions and one boson.

In [13, 14, 15] we developed the technique for construction of a family of spin chain Hamiltonians and their fermionic representations, which have the same energy levels as some basic model($XXZ$, Hubbard, $t - J$ or others) but with huge degeneracy as a result of an affine quantum symmetry added to the basic model. We called this procedure an affinization of the model.

The first example of this type of model was constructed in [16], giving rise to the Hubbard Hamiltonian in the infinite repulsion limit.

In [14] we have fermionized the simplest examples of this newly defined family of models and have shown that it leads to extensions of one-band Hubbard Hamiltonians. The $\eta$-pairing mechanism introduced by Yang [17, 19] was found in one of examples in addition to other exactly solvable Hubbard models with superconducting ground state [20, 21, 22, 23, 24]. The essential property of this extensions is the fact that, besides the ordinary electron hopping and Hubbard interaction terms, they contain also bond-charge interaction, pair-hopping and nearest-neighbour interaction terms. In [15] the $SU(N)$ affinization of $t - J$ model was carried out, giving rise a model where the spin-spin coupling term consists of interaction between the total spins(i.e. the sum of the spins of all band) at nearest-neighbour sites. The presence of the affine symmetry, which ensures the degeneracy of levels exponentially proportional to the length (area) of the space, might lead to a new type of string theory.

In this article we define an extension of $t - J$ model such that an affine quantum group symmetry is present, and we use the Bethe ansatz technique to solve the model. We find the S-matrix excitations on empty background, the ground state and construct the thermodynamic limit of the model. As one might expect, the S-matrix of the excitations on empty background consists of the S-matrix of the ordinary $t - J$ model multiplied by the unity matrix in the additional space of "colours". Therefore the Bethe equations are not different from ones for t-J model, but the rapidities presented in equations correspond to particles with the arbitrary colours. The degeneracy of the corresponding n-particle states come from arbitrary partitions of the colour indices over particles. The same is true for the ground state. The situation is exactly as in gauge theories if we distinguish the states which differs by pure gauge transformations. All this results are presented in the Section 5 of the article.

The thermodynamic limit of the model with corresponding equations are represented

in the Section 6, where we also shown that the S-matrix of our model in an arbitrary background is equal to ordinary $t - J$ model S-matrix multiplied by the Kronecker symbols over the additional colour indices.

## 2 Quantum group invariant Hamiltonians for reducible representations

Let $\mathbf{V} = \oplus_{i=1}^{N} \mathbf{V}_{\lambda_i}$ be a direct sum of finite dimensional irreducible representations $\mathbf{V}_{\lambda_i}$ of quantum group $U_q g$ [32, 33, 34]. We denote by $\mathbf{V}(x_1, \ldots, x_N)$ the representation with spectral parameters $x_i$ of the corresponding affine quantum group $U_q \hat{g}$ [34]:

$$\mathbf{V}(x_1, \ldots, x_N) = \bigoplus_{i=1}^{M} \mathbf{N}_{\lambda_i} \hat{\otimes} \mathbf{V}_{\lambda_i}(x_i), \qquad (1)$$

where all the $\mathbf{V}_{\lambda_i}(x_i)$ are $M$ nonequivalent irreps and $\mathbf{N}_{\lambda_i} \simeq \mathbf{C}^{N_i}$ have dimensions equal to the multiplicity of $\mathbf{V}_{\lambda_i}(x_i)$ in $\mathbf{V}(x_1, \ldots, x_N)$. Note that $\sum_{i=1}^{M} N_i = N$. The $\hat{}$ over the tensor product signifies that $U_q \hat{g}$ does not act on $\mathbf{N}_{\lambda_i} \hat{\otimes} \mathbf{V}_{\lambda_i}(x_i)$ by means of co-multiplication $\Delta$ but instead acts as $\mathrm{id} \otimes g$.

In [14] the general matrix form of the intertwining operator

$$\begin{aligned} H(x_1, \ldots, x_N) : & \\ \mathbf{V}(x_1, \ldots, x_N) \otimes \mathbf{V}(x_1, \ldots, x_N) &\rightarrow \mathbf{V}(x_1, \ldots, x_N) \otimes \mathbf{V}(x_1, \ldots, x_N), \qquad (2) \\ [H(x_1, \ldots, x_N), \Delta(a)] = 0, & \qquad \forall a \in U_q \hat{g} \end{aligned}$$

had been written using the projection operators

$$X_b^a = |a\rangle\langle b| \qquad (3)$$

Here the vectors $|a\rangle$ span the space $\mathbf{V}$. In accordance with the decomposition (1) we will use the double index $a = (n_i, a_i)$, $i = 1, \ldots, M$ where the first index $n_i = 1, \ldots, N_i$ characterises the multiplicity of $\mathbf{V}_{\lambda_i}$ and the second one $a_i = 1, \ldots, \dim \mathbf{V}_{\lambda_i}$ is the vector index of $\mathbf{V}_{\lambda_i}$. Then the intertwining operator (2) is

$$\begin{aligned} H(A, B) = \sum_{i,j=1}^{M} &\left( \sum_{n_i, n_j, m_i, m_j} A_{ij n_i n_j}^{m_i m_j} \sum_{a_i, a_j} X_{(m_i, a_i)}^{(n_i, a_i)} \otimes X_{(m_j, a_j)}^{(n_j, a_j)} \right. \qquad (4) \\ &+ \sum_{n_i, n_j, m_i, m_j} B_{ij n_i n_j}^{m_i m_j} \sum_{a_i, a_j, a_i', a_j'} R_{ij a_i' a_j'}^{a_i a_j}(x_i/x_j) X_{(m_i, a_i)}^{(n_j, a_j')} \otimes X_{(m_j, a_j)}^{(n_i, a_i')} \right), \end{aligned}$$

where the $R$-matrix

$$R_{\mathbf{V}_{\lambda_i} \otimes \mathbf{V}_{\lambda_j}}(x_i/x_j) |a_i\rangle \otimes |a_j\rangle = \sum_{a_i', a_j'} R_{ij a_i' a_j'}^{a_i a_j}(x_i/x_j) |a_i'\rangle \otimes |a_j'\rangle.$$

is the intertwining operator between two actions of affine quantum group $U_q \hat{g}$ on $\mathbf{V}_{\lambda_i} \otimes \mathbf{V}_{\lambda_j}$, which are induced correspondingly by co-multiplication $\Delta$ and opposite co-multiplication $\bar{\Delta}$ [32, 33, 34]:

$$R_{\mathbf{V}_{\lambda_i} \otimes \mathbf{V}_{\lambda_j}}(x_i/x_j) \Delta(g) = \bar{\Delta}(g) R_{\mathbf{V}_{\lambda_i} \otimes \mathbf{V}_{\lambda_j}}(x_i/x_j).$$

$A_{ij}$ and $B_{ij}$, $B_{ii} = 0$ in (4) are arbitrary matrices. In general, $H(A, B)$ depends on deformation parameter $q$ of quantum group, which is included in the $R$-matrix. Note that $R_{\mathbf{V}_{\lambda_i} \otimes \mathbf{V}_{\lambda_j}}(x_i/x_j)$ does not depend on $q$ and is identity only if $\lambda_i$ or $\lambda_j$ are trivial one-dimensional representations. So, in the special case, when the only nontrivial $R$-matrixes in (4) are between representations, one of which is trivial representation, the expression of $H(A, B)$ doesn't depend on $q$. Then $H(A, B)$ commutes with the quantum group action for all values of deformation parameter. In the following we consider only this case.

Following [16, 13] we can from the operator $H$ construct the following Hamiltonian acting on $\mathbf{W} = \mathbf{V}^{\otimes L}$: [6]

$$\mathcal{H} = \sum_{i=1}^{L-1} H_{ii+1}, \tag{5}$$

where the indices $i$ and $i+1$ denote the sites where $H$ acts non-trivially. By the construction, $\mathcal{H}$ is quantum group invariant:

$$[\mathcal{H}, \Delta^{L-1}(g)] = 0, \quad \forall g \in U_q \hat{g}$$

Let us define the projection operators $\mathcal{Q}^i$ on $\mathbf{V}$ for each class of equivalent irreps $(\lambda_i, x_i)$, $i = 1, \ldots, M$

$$\mathcal{Q}^i v_j = \delta_{ij} v_j, \quad \forall v_j \in V_{\lambda_j}(x_j)$$

$$\sum_{i=1}^{M} \mathcal{Q}^i = \mathrm{id}, \quad (\mathcal{Q}^i)^2 = \mathcal{Q}^i$$

Their action on $\mathbf{W}$ is given by

$$\mathcal{Q}^i = \sum_{l=1}^{L} \mathcal{Q}_l^i$$

It is easy to see that these projections commute with Hamiltonian $\mathcal{H}$ and quantum group $U_q \hat{g}$:

$$[\mathcal{Q}^i, \mathcal{H}] = 0, \quad [\mathcal{Q}^i, U_q \hat{g}] = 0 \tag{6}$$

Denote by $\mathbf{W}_{p_1 \ldots p_M}$ the subspace of $\mathbf{W}$ with values $p_i$ of $\mathcal{Q}^i$ on it. Then we have the decomposition

$$\mathbf{W} = \bigoplus_{\substack{p_1, \ldots, p_M \\ p_1 + \ldots + p_M = L}} \mathbf{W}_{p_1 \ldots p_M} \tag{7}$$

Let $\mathbf{V}^0$ be the linear space, spanned by the highest weight vectors in $V$:

$$\mathbf{V}^0 := \oplus_{i=1}^{N} v_{\lambda_i}^0,$$

where $v_{\lambda_i}^0 \in \mathbf{V}_{\lambda_i}$ is a highest weight vector. We also define $\mathbf{W}^0 := \mathbf{V}^{0 \otimes L}$. The space $\mathbf{W}^0$ is $\mathcal{H}$-invariant. For general $q$ the action of $U_q \hat{g}$ on $\mathbf{W}^0$ generate whole space $\mathbf{W}$. Indeed, the $U_q \hat{g}$-action on each state of type $v_{\lambda_{i_1}}^0 \otimes \ldots \otimes v_{\lambda_{i_L}}^0$ generates the space $\mathbf{V}_{\lambda_{i_1}} \otimes \ldots \otimes \mathbf{V}_{\lambda_{i_L}}$, because the tensor product of finite dimensional irreducible representations of an affine quantum group is irreducible [35].

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[6]Here and in the following we omit the dependence on $x_i$

Consider now the subspace $\mathbf{W}^0_{p_1\ldots p_M} = \mathbf{W}^0 \cap \mathbf{W}_{p_1\ldots p_M}$. According to (7) we have the decomposition

$$\mathbf{W}^0 = \bigoplus_{\substack{p_1,\ldots,p_M \\ p_1+\ldots+p_M=L}} \mathbf{W}^0_{p_1\ldots p_M}. \tag{8}$$

Note that

$$d_{p_1\ldots p_M} := \dim \mathbf{W}^0_{p_1\ldots p_M} = \begin{pmatrix} L \\ p_1 \ldots p_M \end{pmatrix} N_1^{p_1} \ldots N_M^{p_M}.$$

Let us define by $\mathcal{H}_0$ the restriction of $\mathcal{H}$ on $\mathbf{W}_0$: $\mathcal{H}_0 := \mathcal{H}|_{\mathbf{W}_0}$. It follows from (6) that Hamiltonians $\mathcal{H}$ and $\mathcal{H}_0$ have block diagonal form with respect to the decompositions (7) and (8), respectively. Every eigenvector $w^0_{\alpha_{p_1\ldots p_M}} \in \mathbf{W}^0_{p_1\ldots p_M}$ with energy value $E_{\alpha_{p_1\ldots p_M}}$ gives rise to an irreducible $U_q\hat{g}$-multiplet $\mathbf{W}_{\alpha_{p_1\ldots p_M}}$ of dimension

$$\dim \mathbf{W}_{\alpha_{p_1\ldots p_M}} = \prod_{k=1}^{M} (\dim \mathbf{V}_{\lambda_k})^{p_k} \tag{9}$$

On $\mathbf{W}_{\alpha_{p_1\ldots p_M}}$ the Hamiltonian $\mathcal{H}$ is diagonal with eigenvalue $E_{\alpha_{p_1\ldots p_M}}$. In particular, in the case when all $\mathbf{V}_{\lambda_i}$ are equivalent, the degeneracy levels are the same for all $E_{\alpha_{p_1\ldots p_M}}$ and are equal to $(\dim \mathbf{V}_\lambda)^L$.

Now, let us assume we know the energy spectrum $E_{\alpha_{p_1\ldots p_M}}$ for $\mathcal{H}_0$. Then the statistical sum is given by

$$Z_{\mathcal{H}_0}(\beta) = \sum_{\substack{p_1,\ldots,p_M \\ p_1+\ldots+p_M=L}} \sum_{\alpha_{p_1\ldots p_M}=1}^{d_{p_1\ldots p_M}} \exp(-\beta E_{\alpha_{p_1\ldots p_M}}), \tag{10}$$

and it follows that the statistical sum of $\mathcal{H}$ has the following form:

$$Z_{\mathcal{H}}(\beta) = \sum_{\substack{p_1,\ldots,p_M \\ p_1+\ldots+p_M=L}} \prod_{k=1}^{M} (\dim \mathbf{V}_{\lambda_k})^{p_k} \sum_{\alpha_{p_1\ldots p_M}=1}^{d_{p_1\ldots p_M}} \exp(-\beta E_{\alpha_{p_1\ldots p_M}}). \tag{11}$$

So, if the underlying Hamiltonian $\mathcal{H}_0$ is integrable and its eigenvectors and eigenvalues can be found, then we know these for $\mathcal{H}$ too. Acting with the quantum group on all eigenvectors of an energy level of $\mathcal{H}_0$ one obtains the whole eigenspace of $\mathcal{H}$ for this level.

# 3 Multi-band $t-J$ model with vanishing spin-spin coupling $J=0$

Let us consider here the quantum group $U_q\widehat{sl}_2$. We choose $\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_j$ for decomposition (1), i.e. we take a direct sum of the trivial spin-0 and the $2j+1$-dimensional spin-$j$ representation of $U_q sl_2$. The $R$-matrix in the second term in (4) does not depend on $q$ and spectral parameters $x_i$ and coincides with the identity, as it was mentioned above. So, using (4) and (5), we obtain the following Hamiltonian

$$\mathcal{H}(t, V_1, V_2) = \sum_{i=1}^{L-1} \left[ -t \sum_{p=1}^{2j+1} (X_{i0}^p X_{i+1p}^0 + X_{i+10}^p X_{ip}^0) + V_1 X_{i0}^0 X_{i+10}^0 \right.$$

$$\left. + V_2 \sum_{p,p'=1}^{2j+1} X_{ip}^p X_{i+1p'}^{p'} \right] \tag{12}$$

The Hamiltonian $\mathcal{H} = \sum_i H_{ii+1}$ was constructed from the operator $H = H_{ii+1}$, where $H$ can be written in the matrix form

$$H = \begin{pmatrix} V_1 & 0 & 0 & 0 \\ 0 & 0 & -t \cdot \mathrm{id} & 0 \\ 0 & -t \cdot \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & V_2 \cdot \mathrm{id} \end{pmatrix}. \tag{13}$$

The projection on the highest weight space coincides with the constructing block of the $XXZ$ Hamiltonian in an external magnetic field. This implies that the restriction of (12) to the space $\mathbf{W}^0$ is

$$\begin{aligned} \mathcal{H}_0(t, W_1, W_2) &= \mathcal{H}_{XXZ}(t, \Delta, B) \\ &= -\frac{t}{2} \sum_{i=1}^{L-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + \frac{B}{2} \sigma_i^z \right), \end{aligned} \tag{14}$$

where

$$\Delta = -\frac{V_1 + V_2}{2t}, \qquad B = \frac{2}{t}(V_1 - V_2) \tag{15}$$

For the special case $V_1 + V_2 = 0$ $\mathcal{H}_0$ gives rise to the free fermionic (or equivalently $XY$) Hamiltonian ($\Delta = 0$).

The projection operators $X_b^a$ are expressed through the fermionic creation-annihilation operators as follows

$$\begin{aligned} X_{i0}^p &= \mathcal{P} c_{i,p}^+, & X_{ip}^0 &= c_{i,p} \mathcal{P}, \\ X_{ip}^p &= n_{i,p} \mathcal{P} = \mathcal{P} n_{i,p}, & X_{i0}^0 &= (1 - n_i) \mathcal{P} = \mathcal{P}(1 - n_i) \end{aligned} \tag{16}$$

Here we introduced the projection operator which forbids double occupation on all sites

$$\mathcal{P} = \prod_{i=1}^L \mathcal{P}_i, \qquad \mathcal{P}_i = \prod_{p \neq p'} (1 - n_{i,p} n_{i,p'})$$

and the total particle number $n_i = \sum_p n_{i,p}$ at site $i$.

After the substitution of the fermionic representation (16) into (12) we obtain

$$\begin{aligned} \mathcal{H}(t, V_1, V_2) = \mathcal{P} \sum_{i=1}^{L-1} \bigg[ &-t \sum_{p=1}^{2j+1} (c_{i,p}^+ c_{i+1,p} + c_{i+1,p}^+ c_{i,p}) + \\ & V n_i n_{i+1} - V_1(n_i + n_{i+1}) + V_1 \bigg] \mathcal{P}, \end{aligned} \tag{17}$$

where $V = V_1 + V_2$. The chemical potential term $-V_1 \sum_{i=1}^{L-1}(n_i + n_{i+1})$ commutes with $\mathcal{H}$ and can be omitted. So, up to unessential boundary and constant terms (17) is a multicomponent $t - J$ model with vanishing spin-spin coupling ($J = 0$)

$$\mathcal{H}(t, V) = \sum_{i=1}^{L-1} \left[ -t \sum_{p=1}^{2j+1} (c_{i,p}^+ c_{i+1,p} + c_{i+1,p}^+ c_{i,p}) + V n_i n_{i+1} \right] + \sum_{i=1}^L \sum_{\substack{p \neq p', \\ p,p'=1}}^{2j+1} U_{p,p'} n_{i,p} n_{i,p'}, \tag{18}$$

where the infinite Hubbard interaction amplitude $U_{p,p'} = +\infty$ between $p$ and $p'$ bands excludes sites with double and more occupations. It follows from the above considerations

that this model has energy levels which coincide with the levels of $XXZ$ Heisenberg model, but that the degeneracy of the levels is different.

For vanishing density-density interaction $V = 0$ the Hamiltonian (18) describes the infinite repulsion limit of the multi-band Hubbard model. Thus, according to (15) $\Delta = 0$ and it has the energy levels of the free fermionic model.

## 4  Multi-band extension of the $t - J$ model with affine quantum group symmetry

In this section we consider Hamiltonians which have the same energy levels as $t - J$ model but have affine quantum group symmetry. Because each site in ordinary $t - J$ model has three states, one should for this purpose take direct sum of three spaces. Let

$$\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_j \oplus \mathbf{V}_j \tag{19}$$

Recall that the $t - J$ model is given by

$$\mathcal{H}_{t-J}(t,J,V) = \mathcal{P} \sum_{i=1}^{L-1} \left[ -t \sum_{\sigma=\pm\frac{1}{2}} (c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma}) + J\mathbf{S}_i\mathbf{S}_{i+1} + Vn_in_{i+1} \right] \mathcal{P}, \tag{20}$$

where $c_\sigma^+$, $c_\sigma$ are creation-annihilation operators of spin-$\frac{1}{2}$ fermion, $\mathbf{S} = \sum_{\sigma,\sigma'} c_\sigma^+ \sigma_{\sigma\sigma'} c_{\sigma'}$ is the fermionic spin operator and $\mathcal{P} = \prod_{i=1}^L (1 - n_{i,\uparrow}n_{i,\downarrow})$ forbids double occupation of sites.

We rewrite it in terms of Hubbard operators $X_b^a$, where $a, b = 0, \pm\frac{1}{2}$:

$$\mathcal{H}(t,J,V) = \sum_{i=1}^{L-1} \left[ \sum_{\sigma=\pm\frac{1}{2}} \left( -t(X_{i0}^\sigma X_{i+1\sigma}^0 + X_{i+10}^\sigma X_{i\sigma}^0) + \frac{1}{2}J \cdot X_{i-\sigma}^\sigma X_{i+1\sigma}^{-\sigma} \right) \right.$$

$$\left. + \sum_{\sigma,\sigma'=\pm\frac{1}{2}} (\sigma\sigma'J + V) X_{i\sigma}^\sigma X_{i+1\sigma'}^{\sigma'} \right] \tag{21}$$

Let us now look at the general expression (4) of intertwining operators $H_{ij}$ acting on the space (19). For convenience we make index change in the following way. The two spin-$j$ representations we use are denoted by $\sigma = \pm\frac{1}{2}$. The intrinsic index in each $V_j^{(\sigma)}$ is denoted by $k$, $k = 1, \ldots, 2j+1$. So, instead of $(n_i, a_i)$ in (4) we have $(\sigma, k)$, if $i$ corresponds to spin-$j$ multiplet. Because the spin-0 singlet is one dimensional and single, we just use for it the index 0. The non-equivalent irreps in (19) are $V_j^{(\sigma)}$ and $V_0$ and, as mentioned above, the $R$-matrix for two such representations is the identity. After performing the first sum in (4) over non-equivalent multiplets we obtain

$$H(A, a, b_1, b_2) = \sum_{\sigma_1,\sigma_2,\sigma_1',\sigma_2'} \left( A_{\sigma_1\sigma_1'}^{\sigma_2\sigma_2'} \sum_{k,k'} X_{(\sigma_2,k)}^{(\sigma_1,k)} \otimes X_{(\sigma_2',k')}^{(\sigma_1',k')} \right) + a \cdot X_0^0 \otimes X_0^0$$

$$+ \sum_{k,\sigma} \left( b_1 \cdot X_0^{(\sigma,k)} \otimes X_{(\sigma,k)}^0 + b_2 \cdot X_{(\sigma,k)}^0 \otimes X_0^{(\sigma,k)} \right) \tag{22}$$

To implement the restriction $H_0(A, a, b_1, b_2)$ of this operator on the highest weight space one just should eliminate the sum over $k, k'$ and put $k = k' = 0$. Comparing (22) and (21) it follows that the expressions coincide if one chooses

$$a = 0 \qquad b_1 = b_2 = -t \qquad A_{\sigma-\sigma}^{-\sigma\sigma} = J/2 \qquad A_{\sigma\sigma}^{\sigma'\sigma'} = (\sigma\sigma') \cdot J + V$$

and choose the other values of $A_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}$ equal zero.

So, the Hamiltonian $\mathcal{H}(A, a, b_1, b_2)$ corresponding to (22) with these values of parameters gives rise to a $t-J$ model (20) on the highest weight space. According to the previous considerations it will have the same energy levels as $t-J$ model, but with different degeneracy. Recall that for $J = 2t$ the $t-J$ model is "supersymmetric" and integrable.

We express the Hubbard operators in terms of multi-band fermionic creation-annihilation operators as follows

$$X_{i0}^{(\sigma,k)} = \mathcal{P}c_{i,\sigma}^{k+}, \qquad X_{i(\sigma,k)}^{0} = c_{i,\sigma}^{k}\mathcal{P},$$

(23)

$$X_{i(-\sigma,k)}^{(\sigma,k)} = c_{i,\sigma}^{k+}c_{i,-\sigma}^{k}\mathcal{P} = \mathcal{P}c_{i,\sigma}^{k+}c_{i,-\sigma}^{k}, \quad X_{i(\sigma,k)}^{(\sigma,k)} = n_{i,\sigma}^{k}\mathcal{P} = \mathcal{P}n_{i,\sigma}^{k}$$

Here as before we used the projection operator, which forbids double occupation on all sites

$$\mathcal{P} = \prod_{i=1}^{L} \mathcal{P}_i, \qquad \mathcal{P}_i = \prod_{(\sigma,k)\neq(\sigma',k')} (1 - n_{i,\sigma}^{k} n_{i,\sigma'}^{k'})$$

Now, we can write down the Hamiltonian (5) in terms of multi-band fermions, substituting (23) into (22). We obtain in this way the multi-band generalization of (20)

$$\mathcal{H}(t, J, V) = \mathcal{P} \sum_{i=1}^{L-1} \left[ -t \sum_{k=1}^{2j+1} \sum_{\sigma=\pm\frac{1}{2}} (c_{i,\sigma}^{k+}c_{i+1,\sigma}^{k} + c_{i+1,\sigma}^{k+}c_{i,\sigma}^{k}) + J\mathbf{S}_i\mathbf{S}_{i+1} + Vn_in_{i+1} \right] \mathcal{P}, \quad (24)$$

Here $k$ is the band index, and $\mathbf{S} = \sum_k \mathbf{S}^k$, $n = \sum_k n^k$ are total spin and total particle number operators. It's easy to see that we have conservation of the particle number operators $\sum_{i,\sigma}^{k} = n^k$ for the all colours $k$.

# 5  Bethe ansatz for the $t-J$ model with affine quantum group symmetry

The goal of this section is to apply the Bethe Ansatz technique to our model and derive the corresponding Bethe equations for the excitations.

After making some trivial Pauli matrix calculations one can represent Hamiltonian (24) as

$$\mathcal{H}(t, J, V) = \mathcal{P} \sum_{i=1}^{L-1} \left[ -t \sum_{k=1}^{2j+1} \sum_{\sigma=\pm\frac{1}{2}} (c_{i,\sigma}^{k+}c_{i+1,\sigma}^{k} + c_{i+1,\sigma}^{k+}c_{i,\sigma}^{k}) + \right.$$

$$\left. 2J \sum_{k,k',\sigma\neq\tau} c_{i,\sigma}^{k+}c_{i,\tau}^{k}c_{i+1,\tau}^{k'+}c_{i+1,\sigma}^{k'} + (V - J)n_in_{i+1} \right] \mathcal{P}. \tag{25}$$

Due to the conservation of the particle number operator $n^k$, and according to the coordinate Bethe Ansatz we look for eigenvectors of (25), corresponding to $N$ fermions of $2j+1$ bands in the following form

$$|\Psi> = \sum_{k_1...k_N=1}^{2j+1} \sum_{x_1\sigma_1} ... \sum_{x_N\sigma_N} \psi^{k_1...k_N}(x_1\sigma_1,...,x_N\sigma_N)c_{x_1\sigma_1}^{k_1+}...c_{x_N\sigma_N}^{k_N+}|0>, \quad (26)$$

where $|0>$ is empty vacuum state.

The eigenvalue equation $H|\Psi> = E|\Psi>$ in the one particle sector

$$-t(\Psi^k(x-1,\sigma) + \Psi^k(x,\sigma)) = E\Psi^k(x,\sigma) \quad (27)$$

gives us

$$E = -2t\cos p \quad (28)$$

after substituting of the plane wave function with momentum p into (27).

The eigenvalue equations in the two particle sector allows us to fix the energy of the state as a sum of two one particle energies, as well as the two two particle scattering matrix $S_{\sigma_1\sigma_2,k_1k_2}^{\sigma_1'\sigma_2',k_1'k_2'}(p_1,p_2)$. We choose the antisymmetric wave function $\Psi^{k_1k_2}(x_1\sigma_1,x_2\sigma_2)$ as

$$\psi^{k_1,k_2}(x_1\sigma_1,x_2\sigma_2) = A^{k_1k_2}(p_1\sigma_1,p_2\sigma_2)e^{i(p_1x_1+p_2x_2)} - A^{k_2k_1}(p_2\sigma_1,p_1\sigma_2)e^{i(p_2x_1+p_1x_2)} \quad (29)$$

for $x_2 \leq x_1$ and

$$\psi^{k_1,k_2}(x_1\sigma_1,x_2\sigma_2) = A^{k_2k_1}(p_2\sigma_2,p_1\sigma_1)e^{i(p_1x_1+p_2x_2)} - A^{k_1k_2}(p_1\sigma_2,p_2\sigma_1)e^{i(p_2x_1+p_1x_2)} \quad (30)$$

for $x_1 \leq x_2$.

The continuity condition at $x_1 = x_2$ should be imposed:

$$A^{k_1k_2}(p_1\sigma_1,p_2\sigma_2) - A^{k_1k_2}(p_2\sigma_1,p_1\sigma_2) = A^{k_2k_1}(p_2\sigma_2,p_1\sigma_1) - A^{k_2k_1}(p_1\sigma_2,p_2\sigma_1) \quad (31)$$

Use of the Hamiltonian (24) and the most general form (26) of the eigenfunctions $\psi$, the eigenvalue equations can be written as

$$-t\left[\psi^{k_1k_2}(x_1+1\sigma_1,x_2\sigma_2)(1-\delta_{x_1+1,x_2}) + \psi^{k_1k_2}(x_1-1\sigma_1,x_2\sigma_2)(1-\delta_{x_1,x_2+1})+\right.$$
$$\left.\psi^{k_1k_2}(x_1\sigma_1,x_2+1\sigma_2)(1-\delta_{x_1,x_2-1}) + t + \psi^{k_1k_2}(x_1\sigma_1,x_2-1\sigma_2)(1-\delta_{x_1-1,x_2})\right] +$$
$$2J\delta_{|x_1-x_2|,1}\psi^{k_1k_2}(x_1\sigma_2,x_2\sigma_1) + +(V-J)\delta_{|x_1-x_2|,1}\psi^{k_1k_2}(x_1\sigma_1,x_2\sigma_2)$$
$$= E\psi^{k_1k_2}(x_1\sigma_1,x_2\sigma_2) \quad (32)$$

The terms $(1-\delta)$ appeared near the hopping terms as a result of projective operator $\mathcal{P}$, preventing double occupancy of the sites.

Two different cases can be considered:

I.  $|x_1 - x_2| > 1$. In this case eigenvalue equation (32) is reduced to

$$-t[\psi^{k_1k_2}(x_1+1\sigma_1,x_2\sigma_2) + \psi^{k_1k_2}(x_1-1\sigma_1,x_2\sigma_2) + \psi^{k_1k_2}(x_1\sigma_1,x_2+1\sigma_2)+$$
$$+\psi^{k_1k_2}(x_1\sigma_1,x_2-1\sigma_2)] = E\psi^{k_1k_2}(x_1\sigma_1,x_2\sigma_2). \quad (33)$$

After substitution of the expression (29)-(30) for the plane waves into (33) and some simple calculations, the spectrum of two particle state can be found.

$$E = -2t(\cos p_1 + \cos p_2). \quad (34)$$

II. $|x_1 - x_2| = 1$. Without loos of generality one can take $x_2 = x_1 + 1$. Then the eigenvalue equation reduces to

$$-t[\psi^{k_1 k_2}(x_1 - 1\sigma_1, x_2\sigma_2) + \psi^{k_1 k_2}(x_1\sigma_1, x_2 + 1\sigma_2) + 2J\psi^{k_1 k_2}(x_1\sigma_2, x_2\sigma_1) +$$
$$+(V - J)\psi^{k_1 k_2}(x_1\sigma_1, x_2\sigma_2) = E\psi^{k_1 k_2}(x_1\sigma_1, x_2\sigma_2) \quad (35)$$

By substitution of (29) and use of continuity conditions (31) one can express the amplitude $A^{k_2 k_1}(p_2\sigma_1, p_1\sigma_2)$ after scattering through $A^{k_1 k_2}(p_1\sigma_1, p_2\sigma_2)$ and $A^{k_1 k_2}(p_1\sigma_2, p_2\sigma_1)$ before, and, therefore get R-matrix of the model:

$$A^{k_2 k_1}(p_2\sigma_1, p_1\sigma_2) = R^{\sigma_1'\sigma_2', k_2 k_1}_{\sigma_1\sigma_2, k_1' k_2'} A^{k_1' k_2'}(p_1\sigma_1', p_2\sigma_2') \quad (36)$$

where R-matrix is the product of ordinary $t - J$ model R-matrix multiplied the permutation operator in the $k$-index space.

$$R^{\sigma_1'\sigma_2', k_2 k_1}_{\sigma_1\sigma_2, k_1' k_2'} = R^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(t - J) \cdot \delta^{k_2}_{k_1'} \cdot \delta^{k_1}_{k_2'}. \quad (37)$$

At the supersymmetric point $2J = t$, $V = -J/4$

$$R^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(t - J) = \frac{(\lambda_1 - \lambda_2)\hat{P} + i\hat{I}}{\lambda_1 - \lambda_2 + i}. \quad (38)$$

In (38) $\lambda = \frac{1}{2}\cot\frac{p}{2}$ is the rapidity, $\hat{I} = \delta^{\sigma_1'}_{\sigma_1}\delta^{\sigma_2'}_{\sigma_2}$ ,and $\hat{P} = \delta^{\sigma_2'}_{\sigma_1}\delta^{\sigma_1'}_{\sigma_2}$.

The scattering matrix $S^{\sigma_1'\sigma_2', k_1' k_2'}_{\sigma_1\sigma_2, k_1 k_2}$ will be defined multiplying the R-matrix by permutation operator $\hat{P}$ in the spin($\sigma$) and colour($k$) spaces.

The exact integrability of the model is connected with the fact that the S-matrix should fulfill the Yang-Baxter triangular relations

$$S^{\sigma_1'\sigma_2', k_1' k_2'}_{\sigma_1\sigma_2, k_1 k_2}(\lambda_1 - \lambda_2) \cdot S^{\sigma_1''\sigma_3', k_1'' k_3'}_{\sigma_1'\sigma_3, k_1' k_3}(\lambda_1 - \lambda_3) \cdot S^{\sigma_2''\sigma_3'', k_2'' k_3''}_{\sigma_2'\sigma_3', k_2' k_3'}(\lambda_2 - \lambda_3) =$$
$$S^{\sigma_2'\sigma_3', k_2' k_3'}_{\sigma_2\sigma_3, k_2 k_3}(\lambda_2 - \lambda_3) \cdot S^{\sigma_1'\sigma_3'', k_1' k_3''}_{\sigma_1\sigma_3', k_1 k_3'}(\lambda_1 - \lambda_3) \cdot S^{\sigma_1''\sigma_3'', k_1'' k_2''}_{\sigma_1'\sigma_2', k_1' k_2'}(\lambda_1 - \lambda_2) \quad (39)$$

and constraints on $t, J, V$ are imposed just by these equations.

Consider now $N$ itinerant electrons in a box of $L$ sites with periodic boundary conditions. If we successively make change of position of an electron and its neighbouring electron in a chain, each interchange produces a scattering matrix and when the particle comes back to its starting position from the other side, we will have the cyclic product of S-matrices, which is called the Transfer Matrix:

$$\hat{T}_j(\lambda_j) = \hat{S}_{j,j+1}(\lambda_j - \lambda_{j+1})....\hat{S}_{j,N}(\lambda_j - \lambda_N) \cdot \hat{S}_{j,1}(\lambda_j - \lambda_1)....\hat{S}_{j,j-1}(\lambda_j - \lambda_{j-1}) \quad (40)$$

Here we skip the matrix indexes $\sigma$ and $k$ while the hat on $S$ means the operator in that space. The periodicity means that the Transfer Matrix has to be diagonal for all $j = 1.....N$ with the eigenvalues $exp(ip_j N)$, or, on terms of rapidity $\lambda_j$

$$e^{ip_j N} = \left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^L \quad (41)$$

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The matrix (40) is the trace of so called monodromy matrix, which by definition is the product of the $S$-matrices without taking trace. Hence, the monodromy matrix can be considered as a $(2 \times 2) \otimes ((2j + 1) \times (2j + 1))$ matrix in the spin $\sigma$ and colour $k$ spaces. Let's remember now that this operator is a unity operator in the colour space. We will not describe here the details of the algebraic Bethe Ansatz(see e.g. [30]), but already now it is clear, that because the monodromy matrix of our model is the monodromy matrix of the ordinary $t - J$ model multiplied by the unity matrix in the $k$-space, the generalization of the algebraic Bethe Ansatz to the present model give rise to the same equations as the ordinary $t - J$ model.

Specifically we get:

$$\left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^L = \prod_{\beta=1}^{M} \frac{\lambda_j - \Lambda_\beta + i/2}{\lambda_j - \Lambda_\beta - i/2}, \qquad j = 1, ... N \tag{42}$$

$$\prod_{j=1}^{N} \frac{\Lambda_\alpha - \lambda_j + i/2}{\Lambda_\alpha - \lambda_j - i/2} = - \prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + i}{\Lambda_\alpha - \Lambda_\beta - i}, \qquad \alpha = 1, ... M \tag{43}$$

where $L$ is the number of lattice sites, $N$-is the number of electrons and $M$ is the number of spin down electrons.

We see that colour is disappeared from the equations, which means that we can make arbitrary partition of colour charges on the state of $N$ particles in a chain and all wave vectors will become eigenvalues of the Hamiltonian, provided that their $\lambda$'s fulfill the Bethe equations.

Equations (42)-(43) are Lai's [8] form of Bethe equations, written on a basis of empty background.

The total energy is given by

$$E = -2N + 2 \sum_{j=1}^{N} \frac{1/2}{\lambda_j^2 + 1/4} \tag{44}$$

However for the construction of the thermodynamic limit of our model it is more convenient to use the Sutherland's form [7] of the Bethe equations, which is equivalent to Lai's equations [11, 25]. In Sutherland's representation one start with a ferromagnetic pseudo-vacuum state with all spins up and consider excitations as $N_h$-holes(empty sites) and $M$ spin down electrons.

In Sutherland's representation we have

$$\left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^L = \prod_{\beta=1}^{M} \frac{\lambda_j - \Lambda_\beta + i/2}{\lambda_j - \Lambda_\beta - i/2}, \qquad j = 1, ... M + N_h$$

$$1 = \prod_{j=1}^{M+N_h} \frac{\lambda_j - \Lambda_\beta + i/2}{\lambda_j - \Lambda_\beta - i/2}, \qquad \beta = 1, ... M \tag{45}$$

The equations (45) have real and complex solutions. The complex solutions are in a form known as strings, which may be found by fixing $N_h$ and $M$ (they are conserved quantities) and letting the lattice size $L \to \infty$. Following [11, 26] one finds complex solutions in the form

$$\lambda_j = \lambda + i(n + 1 - 2j), \qquad j = 1, ... n$$
$$\Lambda_\tau = \lambda + i(n - 2\tau) \qquad \tau = 1, ... n - 1, \tag{46}$$

for arbitrary $n$. In a finite box these string solutions are not exact but as in [8, 28, 29] we will assume that the corrections are small in power of $L$ and hence vanish in the thermodynamic limit.

We would like to write the equation for the centrum of the strings of length $n$, $\lambda_\alpha^n$, (which are the real variables), let their number is $M'$ and rapidities $\Lambda_\beta$ of spin down electrons.

Substituting (45) into Lai's form of Bethe equations and taking the logarithms from the left and right hand sides, one can get

$$L\theta(\frac{\lambda_\alpha^n}{n}) = 2\pi I_\alpha^n + \sum_{m=1}^{\infty} \sum_{\gamma}^{N_m} \Theta_{nm}(\lambda_\alpha^n - \lambda_\gamma^m) + \sum_{\beta=1}^{M} \theta(\frac{\lambda_\alpha^n - \Lambda_\beta}{n}) \tag{47}$$

and

$$\sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} \theta(\frac{\Lambda_\beta - \lambda_{lpha}^n}{n}) = 2\pi J_\beta \tag{48}$$

where $N_m$ is the number of strings of length $m$, $\theta(x) = 2tan^{-1}(x)$ and

$$\Theta_{mn} = \begin{cases} \theta(\frac{x}{|n-m|}) + 2\theta(\frac{x}{|n-m|+2}) + ... + 2\theta(\frac{x}{n+m-2}) & if \ n \neq m \\ 2\theta(\frac{x}{2}) + 2\theta(\frac{x}{4}) + ... + 2\theta(\frac{x}{n+m-2}) & if \ n = m \end{cases} \tag{49}$$

$I_\alpha^n$ and $J_\beta$ are integers(half integers) appearing after choice of branch of the logarithm. The integers $M$, $N_m$ and $M'$ satisfy the relation

$$M' = M + \sum_{m=1}^{\infty} mN_m \tag{50}$$

Any solution is defined by a set of $I_\alpha^n$ and $J_\beta$. All possible integer(half integer) values defines the states called vacancies. The vacancies may be occupied by particles of colour $k$ or not occupied at all. In a case of occupied vacancies one can mark the corresponding $I_\alpha^n$ and $J_\beta$ by colour index $k$ as for particles, $I_\alpha^{n,k}$ and $J_\beta^k$. When the state is empty, the corresponding integer will be mentioned as $\bar{I}_\alpha^n$ and $\bar{J}_\beta$.

One may calculate the number of string states (46) taking account also colour degrees of freedom and will get $(2j + 1)^L$.

## 6 The thermodynamic limit of the Bethe Ansatz equations

In the thermodynamic limit $L \to \infty$ the solution becomes densely packed (the difference of two neighbour solutions is of order $1/L$, $\lambda_{j+1}^n - \lambda_j^n = O(1/L)$) and one can introduce density functions of the states and pass from sum's to integrals in the equations (47)-(48).

Let's define the particle and hole densities as follows:

$$\rho_p^{n,k}(\lambda) = \lim_{L\to\infty} \frac{1}{L(\lambda_{I_{j+1}^{n,k}} - \lambda_{I_j^{n,k}})}, \qquad \rho_h^n(\lambda) = \lim_{L\to\infty} \frac{1}{L(\lambda_{\bar{I}_{j+1}^n} - \lambda_{\bar{I}_j^n})}$$

$$\sigma_p^k(\lambda) = \lim_{L\to\infty} \frac{1}{L(\Lambda_{J_{j+1}^k} - \Lambda_{J_j^k})}, \qquad \sigma_h(\lambda) = \lim_{L\to\infty} \frac{1}{L(\Lambda_{\bar{J}_{j+1}} - \Lambda_{\bar{J}_j})} \tag{51}$$

and the sum of the hole and particle densities defines the density functions of vacancies.

$$\rho_t^n = \sum_{k=1}^{2j+1} \rho_p^{n,k} + \rho_h^n, \qquad \sigma_t = \sum_{k=1}^{2j+1} \sigma_p^k + \sigma_h. \tag{52}$$

The passage from sum to integral is straightforward and in the thermodynamic limit the Bethe equations (47)-(48) become integral equations:

$$\rho_t^n(\lambda) = f_n(\lambda) - \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\lambda' A_{n,m}(\lambda - \lambda') \rho_p^m(\lambda') - \int_{-\infty}^{\infty} d\Lambda f_n(\lambda - \Lambda) \sigma_p(\Lambda) \tag{53}$$

and

$$\sigma_t(\Lambda) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\lambda f_n(\Lambda - \lambda) \rho_p^n(\lambda) \tag{54}$$

$f_n(\lambda)$ and $A_{n,m}(\lambda)$ in equations (53)-(54) are defined by

$$f_n(\lambda) = \frac{1}{2\pi} \frac{d\theta(\lambda)}{d\lambda} = \frac{1}{\pi} \frac{n}{\lambda^2 + n^2} \tag{55}$$

and

$$A_{n,m} = \begin{cases} f_{|n-m|}(\lambda) + 2f_{|n-m|+2}(\lambda) + ... + 2f_{n+m-2}(\lambda), & if \quad n \neq m \\ 2f_2(\lambda) + 2f_4(\lambda) + ... + 2f_{2n-2}(\lambda), & if \quad n = m \end{cases} \tag{56}$$

Following Yang and Yang [27], Takahashi [28] and Gaudin [29], we can write down the thermodynamic equilibrium equations. For the ordinary $t - J$ model it was done in [5, 31, 26].

The conserved quantities of our model are:

- *the energy* (the expression in ferromagnetic background slightly differs from the Lai's expression (44))

$$E = L \left( 1 - 4\pi \sum_{n=1}^{\infty} \sum_{k=1}^{2j+1} \int d\lambda f_n(\lambda) \rho_p^{n,k}(\lambda) \right), \tag{57}$$

- *the number of different particles*

$$N^k = L \left( 1 - \int d\lambda \sum_{n=1}^{\infty} \rho_p^{n,k}(\lambda) + \int d\Lambda \sigma_p^k(\Lambda) \right) \tag{58}$$

- and *the magnetisation*

$$S_z = \frac{L}{2} \left( 1 - \sum_{n=1}^{\infty} \sum_{k=1}^{2j+1} (2n - 1) \int d\lambda \rho_p^{n,k}(\lambda) - \sum_{k=1}^{2j+1} \int d\Lambda \sigma_p^k(\Lambda) \right). \tag{59}$$

Correspondingly, for the free energy $F$ one can write

$$F = E - \sum_{k=1}^{2j+1} \mu_k N^k - B S_z - T S, \tag{60}$$

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where $\mu_k$ are the chemical potentials of particles, $B$ is the external magnetic field and $T$ is the temperature. We will calculate the entropy $S$ in a standard way, as the logarithm of the number of possible states of the model in the interval $d\lambda$.

$$N(\lambda, d\lambda) = e^{S(\lambda)d\lambda} = \frac{L(\sum_{k=1}^{2j+1} \rho_p^k d\lambda)!}{\prod_{k=1}^{2j+1}(L\rho_p^k d\lambda)!(L\rho_h d\lambda)!} \tag{61}$$

By Stirling's formula one finds from (61)

$$S = L \left\{ \int d\lambda \sum_{m=1}^{\infty} \left[ \rho_t^n \, log\rho_t^n - \sum_{k=1}^{2j+1} \rho_p^{n,k} \log \rho_p^{n,k} - \rho_h^n \log \rho_h^n \right] + \right.$$
$$\left. + \int d\Lambda \left[ \sigma_t \log \sigma_t - \sum_{k=1}^{2j+1} \sigma_p^k \log \sigma_p^k - \sigma_h \log \sigma_h \right] \right\}. \tag{62}$$

To obtain the equilibrium equations one should minimise the free energy over density functions of the particles and holes. Puting $\delta F = 0$ for variations of the density functions $\rho_p^{n,k}, \rho_h^n, \sigma_p^k, \sigma_h$ one gets after some algebraic transformations an infinite set of integral equations for the densities. If one defines the excited energies as usual

$$\frac{\rho_p^{n,k}}{\rho_t^n - \rho_p^{n,k}} = e^{-\frac{\epsilon^{n,k}}{T}}, \qquad \frac{\sigma_p^k}{\sigma_t - \sigma_p^k} = e^{-\frac{\epsilon_\Lambda^k}{T}} \tag{63}$$

and

$$\epsilon_0^{n,k} = -4\pi f_n(\lambda) - (2n-1)B + \mu_k \tag{64}$$

we obtain

$$\epsilon^{n,k}(\lambda) = \epsilon_0^{n,k} + T \sum_{m=1}^{\infty} \int d\lambda' A_{n,m}(\lambda - \lambda') \log\left(1 + e^{\frac{-\epsilon^{m,k}(\lambda')}{T}}\right) -$$
$$- T \int d\Lambda f_n(\lambda - \Lambda) \log\left(1 + e^{\frac{-\epsilon_\Lambda^k(\Lambda)}{T}}\right) \tag{65}$$

for $n = 1, .... \infty$, and

$$\epsilon_\Lambda^k = -\mu_k - B + T \sum_{n=1}^{\infty} \int d\lambda f_n(\lambda - \Lambda) \log\left(1 + e^{\frac{-\epsilon^{n,k}(\lambda)}{T}}\right). \tag{66}$$

It is seen from these equations for densities and excitation energies that if all particles have the same chemical potential $\mu = \mu_k$, the solutions will be independent of $k$. Correspondingly, for minimum value of the free energy one can has

$$\frac{F}{L} = (1 + B - \mu) - T \sum_{m=1}^{\infty} \int d\lambda f_m(\lambda) \log\left(1 + e^{\frac{-\epsilon^m(\lambda)}{T}}\right) =$$
$$= (1 - 2\mu) - \epsilon_\Lambda(\Lambda = 0) \tag{67}$$

in correspondence with [5, 26]

In order to analyse the ground state further one should take the limit $T \to 0$ in equations (65)-(66) and then put the solution obtained into the expression for the energy (57). If we suppose that $\epsilon^n > 0$ for $n > 1$ (which can be checked after all by analysing the corresponding

equations for the $\epsilon^n$'s) in the sum of the integral equations (65)-(66), only one term will contribute in the limit $T \to 0$, and we have

$$\epsilon^{1,k}(\lambda) = \epsilon_0^{1,k} + \int d\Lambda f_1(\lambda - \Lambda)\bar{\epsilon}_\Lambda^k(\Lambda),$$

$$\epsilon_\Lambda^k(\Lambda) = -(\mu_k + B) - \int d\lambda f_1(\Lambda - \lambda)\bar{\epsilon}^{1,k}(\lambda), \tag{68}$$

where

$$\bar{\epsilon}_\Lambda^k(\Lambda) = \begin{cases} \epsilon_\Lambda^k(\Lambda), & if \ \epsilon_\Lambda^k(\Lambda) < 0 \\ 0 & if \ \epsilon_\Lambda^k(\Lambda) > 0 \end{cases}, \quad \bar{\epsilon}^{1,k}(\lambda) = \begin{cases} \epsilon^{1,k}(\lambda), & if \ \epsilon^{1,k}(\lambda) < 0 \\ 0 & if \ \epsilon_\Lambda^k(\Lambda) > 0 \end{cases}. \tag{69}$$

Hence, in Sutherland's approach we are lead to the concept of two seas: one for real(non-string) solutions, another for rapidity $\Lambda$ (spin down electrons).

Suppose that $\epsilon^{1,k}(\lambda) = 0$ at some point $\lambda = Q$ and $\epsilon_\Lambda^k = 0$ at $\Lambda = Q_\Lambda$, which are the Fermi rapidities. Then we can write the equations for the ground state energy as

$$\frac{E_0}{L} = 1 - 4\pi \sum_{k=1}^{2j+1} \int_{-Q}^{Q} d\lambda f_1(\lambda)\rho_p^{1,k}(\lambda). \tag{70}$$

The equations for the density functions also simplify

$$\sum_{k=1}^{2j+1} \rho_p^{1,k}(\lambda) = f_1(\lambda) - \sum_{k=1}^{2j+1} \left( \int_{-\infty}^{-Q_\Lambda} + \int_{Q_\Lambda}^{\infty} \right) d\Lambda f_1(\lambda - \Lambda)\sigma_p^k(\Lambda) \tag{71}$$

and

$$\sum_{k=1}^{2j+1} \sigma_p(\Lambda) = \sum_{k=1}^{2j+1} \int_{-Q}^{Q} d\lambda f_1(\Lambda - \lambda)\rho_p^{1,k}(\lambda) \tag{72}$$

The Fermi boundaries $Q$ and $Q_\Lambda$ are determined from the particle number and magnetisation equations

$$\frac{N}{L} = 1 - \sum_{k=1}^{2j+1} \left[ \int_{-Q}^{Q} d\lambda \rho_p^{1,k}(\lambda) + \left( \int_{-\infty}^{-Q_\Lambda} + \int_{Q_\Lambda}^{\infty} \right) d\Lambda \sigma_p^k(\Lambda) \right],$$

$$\frac{2S_z}{L} = 1 - \sum_{k=1}^{2j+1} \left[ \int_{-Q}^{Q} d\lambda \rho_p^{1,k}(\lambda) + \left( \int_{-\infty}^{-Q_\Lambda} - \int_{Q_\Lambda}^{\infty} \right) d\Lambda \sigma_p^k(\Lambda) \right]. \tag{73}$$

Due to a theorem by Lieb and Mattis [36], in zero magnetic field $S_z$ should be zero, which means that spin down electrons constitute half of all electrons $M = N/2$ and there is not string states in the ground state.

We see that the equations for the ground state are invariant under transformation of the index $k$, which means that we can arbitrarily distribute the colour over the particles. They are as gauge modes in the vacuum of gauge theories.

In the half filled case $Q_\Lambda = 0$ and $Q = \infty$ (as easy to see in Lai's form of equations [5]) and following [5, 9] one can obtain $E_0 = 1 - 2\log 2$.

It is also possible , following [26], to consider the excitations of the model, introduce the so called shift functions and write down equations for them. After that, following [37], one can find the $S$-matrix for the excitations of the model. It follows as above, that the $S$-matrix of our model is equal to the $S$-matrix of ordinary $t - J$ model multiplied by unity operators in additional colour space.

# 7 Conclusions

We have constructed a one dimensional model which is based on the t-J model of strongly correlated electrons, but which has an additional quantum group symmetry, ensuring the degeneration of the states. We use the Bethe Ansatz technique to investigate this model. The equations for density functions, written in the thermodynamic limit , demonstrate that the additional degrees of freedom of the model behave as gauge modes. The presence of these modes, in our opinion, gives rise a possibility to construct a new type of integrable models, if one has an interaction between two models of this kind. Also, different topological properties, usually appearing in gauge theories, could be present in our model, and would be interesting to investigate.

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